

An analysis of the Isgur-Wise Function and its derivatives within a Heavy-Light QCD Quark Model

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Abstract

In determining the mesonic wave function from QCD inspired potential model, if the linear confinement term is taken as parent (with columbic term as perturbation), Airy's function appears in the resultant wave function - which is an infinite series. In the study of Isgur-Wise function (IWF) and its derivatives with such a wave function, the infinite upper limit of integration gives rise to divergence. In this paper, we have proposed some reasonable cut-off values for the upper limit of such integrations and studied the subsequent effect on the results. We also study the sensitivity of the order of polynomial approximation of the infinite Airy series in calculating the derivatives of IWF.

Key words : Isgur-Wise Function, Airy's function, Dalgarno's method.

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1 Introduction:

The quark composite systems with one heavy quark have been the focus of interest for the last few decades. In infinite heavy quark mass limit, the heavy quark sector of QCD becomes independent of quark masses and the effective Lagrangian of the heavy quark effective theory (HQET) exhibits additional spin flavour symmetries and this simplifies the calculations of matrix elements of electroweak transitions. The semi-leptonic transitions between heavy mesons are of great importance. In the limit of infinite quark masses, all the mesonic form factors can be expressed in terms of a single universal function, called the Isgur-Wise Function [1]. This IWF, the universal function describing a class of semi-leptonic transitions, has been studied for many years. The shape (or form) of the IWF and its derivatives (slope and curvature) at zero recoil, is essential for determination of CKM matrix elements and that requires a reasonable description of the IWF. The main part of the IW function is the wave function of the hadrons. The wave function for the heavy-light mesons have been calculated earlier within the framework of QCD potential model (Cornell potential) [2,3] with considerable accuracy [4]. This has been deduced both with columbic term in potential as parent [5] and also with the linear confinement term as parent [6]. The characteristics like slope (charge radii) and curvature (convexity parameter) of IWF have been reported in both the two cases with certain limitations. In the present paper, we have considered the linear confinement term (**br**) as parent and Columbic term as perturbation. This has been done applying Dalgarno's method of perturbation theory up to first order correction [7]. As Airy's function appears in the wave function for unperturbed Hamiltonian, the integrability of the otherwise divergent infinite Airy's function series in the IWF is a question of consideration in the present work. We have introduced some reasonable cut-off for upper limit of integration of IWF keeping in mind the nature of Airy's function and the boundary condition of IWF ($\xi(1)=1$) and studied the variation of results with this

cut-off value at different orders of Airy's function. We also study the sensitivity of the order of polynomial approximation of the infinite Airy's function when compared with experimental result of the derivatives of IWF. The section 2 contains the essential formalism, section 3 , the calculations and results and the section 4 contains conclusion.

2 Formalism:

2.1 Potential Model:

The potential under consideration (Cornell potential) is :

$$V(r) = -\frac{4\alpha_s}{3r} + br + c \quad (1)$$

Here we take \mathbf{br} as parent so that our unperturbed Hamiltonian [8] is

$$H_0 = -\frac{\nabla^2}{2\mu} + br \quad (2)$$

with

$$H' = -\frac{4\alpha_s}{3r} + c \quad (3)$$

as perturbation. Here μ is the reduced mass, which is

$$\mu = \frac{m_q m_Q}{m_q + m_Q} \quad (4)$$

We take the value of b to be 0.183 GeV^2 from charomonium spectroscopy [9] and constant C to be 1 GeV . It is to be mentioned that, In the infinite heavy quark mass limit ($m_Q \rightarrow \infty$) ,

$$\mu = \lim_{m_Q \rightarrow \infty} \frac{m_q m_Q}{m_q + m_Q} \approx m_q \quad (5)$$

Under this consideration, the two body Schrodinger equation [11, 12] for the Hamiltonian $H = H_0 + H'$ is :

$$H|\Psi\rangle = (H_0 + H')|\Psi\rangle = E|\Psi\rangle \quad (6)$$

2.2 Wave Function:

To find the unperturbed wave function corresponding to H_0 we get the radial equation for potential br for ground state S ($l = 0$), following the formalism of reference [13] , as :

$$\left[-\frac{1}{2\mu}\left(\frac{d^2}{dr^2} + \frac{2}{r}\frac{d}{dr}\right) + \sigma r\right] = ER(r) \quad (7)$$

where $R(r)$ is the radial wave function. We introduce $u(r) = r R(r)$ and dimensionless variable ρ , where -

$$\rho = (2\mu b)^{1/3} r - \left(\frac{2\mu}{b^2}\right)^{1/3} E \quad (8)$$

The above equation then reduces to:

$$\frac{d^2 u}{d\rho^2} - \rho u = 0 \quad (9)$$

The solution of this second order homogeneous differential equation [14] contains linear combination of two types of Airy's functions $Ai[r]$ and $Bi[r]$. The nature of the Airy's function [15,16] reveals that

$$\begin{aligned} Ai[r] &\rightarrow 0 \text{ as } r \rightarrow \infty \text{ and} \\ Bi[r] &\rightarrow \infty \text{ as } r \rightarrow \infty. \end{aligned}$$

So, it is reasonable to reject the $Bi[r]$ part of the solution. The radial wave function has thus the form:

$$u(r) = N Ai[(2\mu b)^{1/3}(r - \frac{E}{b})] \quad (10)$$

where N is our normalization constant which has the dimension of $GeV^{1/2}$. The boundary condition $u(0) = 0$ [17] gives us the unperturbed energy for ground state [13]:

$$W^{(0)} = E = -(\frac{b^2}{2\mu})^{1/3} \rho_0 \quad (11)$$

where ρ_0 is the zero of the $Ai[r]$, such that $Ai[\rho_0] = 0$ [18]. ρ_0 has the explicit form -

$$\rho_0 = -[\frac{3\pi(4n-1)}{8}]^{2/3}, \text{ in our case } n = 1 \text{ for ground state.} \quad (12)$$

From this we get the unperturbed wave function for ground state to be :

$$\Psi^0(r) = \frac{N}{2\sqrt{\pi}r} Ai[(2\mu b)^{1/3}r + \rho_0] \quad (13)$$

$$\Psi^0(r) = \frac{N}{2\sqrt{\pi}r} Ai[\rho_1 r + \rho_0] \quad (14)$$

where we have taken $\rho_1 = (2\mu b)^{1/3}$.

The first order perturbed eigen function $\Psi^{(1)}$ and eigen energy W^1 can be calculated using the following relation:

$$H_0 \Psi' + H' \Psi^0 = W^{(0)} \Psi' + W^{(1)} \Psi^0 \quad (15)$$

We find,

$$W^{(1)} = \int_0^\infty r^2 H' | \Psi^0 |^2 dr \quad (16)$$

Employing Dalgarno's method [19], the first order wave function comes out to be [6]:

$$\Psi^{(1)}(r) = -\frac{4\alpha_s}{3}(\frac{a_0}{r} + br + c) \quad (17)$$

where a_0, a_1 and a_2 are terms which involve $\alpha_s, b, \mu, W^{(1)}, E$ and c . These are having dimensions of $GeV^{1/2}, GeV^{3/2}, GeV^{5/2}$ respectively and have explicit form, considering Airy order up to r^3 , as given below [6].

$$a_0 = \frac{0.8808(b\mu)^{1/3}}{(E-c)} - \frac{a_2}{\mu(E-c)} + \frac{4W^1 \times 0.21005}{3\alpha_s(E-c)} \quad (18)$$

$$a_1 = \frac{ba_0}{(E-c)} + \frac{4W^1 \times 0.8808(b\mu)^{1/3}}{3\alpha_s(E-c)} - \frac{0.6535 \times (b\mu)^{2/3}}{(E-c)} \quad (19)$$

$$a_2 = \frac{4\mu W^1 \times 0.1183}{3\alpha_s} \quad (20)$$

The total wave function with first order correction is:

$$\Psi_{tot}(r) = \Psi^0(r) + \Psi^1(r) \quad (21)$$

which upon substitution yields-

$$\Psi_{tot}(r) = N' \left[\frac{N}{2\sqrt{\pi}r} A_i[(\rho_1 r + \rho_0)] - \frac{4\alpha_s}{3} \left(\frac{a_0}{r} + br + c \right) \right] \quad (22)$$

Here N' is the normalization constant of total wave function which is also having the dimension of $GeV^{1/2}$. Considering relativistic effect on the wave function, the total relativistic wave function is given by [11]:

$$\Psi_{rel}(r) = \Psi_{tot}(r) \left(\frac{r}{a_b} \right)^{-\epsilon} \quad (23)$$

This gives,

$$\Psi_{rel}(r) = N' \left[\frac{N}{2\sqrt{\pi}r} A_i[(\rho_1 r + \rho_0)] - \frac{4\alpha_s}{3} \left(\frac{a_0}{r} + br + c \right) \right] \left(\frac{r}{a_b} \right)^{-\epsilon} \quad (24)$$

Here,

$$a_b = \frac{4}{3\mu\alpha_s} \text{ and } \epsilon = 1 - \sqrt{1 - \left(\frac{4\alpha_s}{3} \right)^2} \quad (25)$$

2.3 Isgur-Wise Function:

In case of semi-leptonic decay of hadrons (mesons), in the infinite mass limit, a new symmetry called spin-flavored symmetry, will emerge and the Heavy Quark Effective Theory (HQET) will be suitable. In this theory, the strong interactions of the heavy quarks are independent of its spin and mass[20] and all the form factors are completely determined, at all momentum transfers, in terms of only one elastic form factor function, the universal Isgur-Wise function $\xi(v, v')$. $\xi(v, v')$ depends only upon the four velocities v_ν and $v_{\nu'}$ of heavy particle before and after decay. This $\xi(v, v')$ is normalized at zero recoil [21]. If $y = v_\nu \cdot v_{\nu'}$, then, for zero recoil ($y=1$), $\xi(y) = 1$. With increasing recoil y grows. In explicit form IW function can be expressed as :

$$\xi(y) = 1 - \rho^2(y-1) + C(y-1)^2 + \dots \quad (26)$$

ρ^2 is the slope parameter at $y=1$ given by -

$$\rho^2 = - \frac{\delta \xi(y)}{\delta y} \Big|_{y=1} \quad (27)$$

ρ is known as the charge radius.

C is the convexity parameter given by -

$$C == \frac{\delta^2 \xi(y)}{\delta y^2} \Big|_{y=1} \quad (28)$$

The calculation of this IW function is non-perturbative in principle and is performed for different phenomenological wave functions of mesons [22]. This function depends upon the meson wave function and some kinematic factor, as given below :

$$\xi(y) = \int_0^\infty 4\pi r^2 |\Psi(r)|^2 \cos(pr) dr \quad (29)$$

where $\cos(pr) = 1 - \frac{p^2 r^2}{2} + \frac{p^4 r^4}{24} + \dots$ with $p^2 = 2\mu^2(y-1)$. Taking $\cos(pr)$ up to $O(r^4)$ we get,

$$\xi(y) = \int_0^\infty 4\pi r^2 |\Psi(r)|^2 \left(1 - \frac{p^2 r^2}{2} + \frac{p^4 r^4}{24}\right) dr \quad (30)$$

$$= \int_0^\infty 4\pi r^2 |\Psi(r)|^2 dr - \int_0^\infty 4\pi r^2 |\Psi(r)|^2 \left[2\mu^2 \frac{(y-1)r^2}{2}\right] dr + \int_0^\infty 4\pi r^2 |\Psi(r)|^2 \left[4\mu^4 \frac{(y-1)^2 r^4}{24}\right] dr \quad (31)$$

$$= \int_0^\infty 4\pi r^2 |\Psi(r)|^2 dr - [4\pi\mu^2 \int_0^\infty r^4 |\Psi(r)|^2 dr](y-1) + \left[\frac{2}{3}\pi\mu^4 \int_0^\infty r^6 |\Psi(r)|^2 dr\right](y-1)^2 \quad (32)$$

Equations (26) and (32) give us :

$$\rho^2 = [4\pi\mu^2 \int_0^\infty r^4 |\Psi(r)|^2 dr] \quad (33)$$

$$C = \left[\frac{2}{3}\pi\mu^4 \int_0^\infty r^6 |\Psi(r)|^2 dr\right] \quad \text{and} \quad (34)$$

$$\int_0^\infty 4\pi r^2 |\Psi(r)|^2 dr = 1 \quad (35)$$

Equation (35) gives the normalization constants N and N' for $\Psi^0(r)$ and $\Psi_{tot}(r)$ respectively , as :

$$N = \frac{1}{(\int_0^\infty A_i[(\rho_1 r + \rho_0)]^{1/2} dr)} \quad (36)$$

$$N' = \frac{1}{[4\pi \int_0^\infty r^2 (\frac{1}{2\sqrt{\pi}r} A_i(\rho_1 r + \rho_0) - \frac{4\alpha_s}{3} (\frac{a_0}{r} + br + c))]^{1/2}} \quad (37)$$

3 Calculation and result:

With linear confinement term of potential as parent, the wave functions contain Airy's function $Ai[\rho_1 r + \rho_0]$, which is an infinite series in itself [24].

$$\begin{aligned} A_i[\rho_1 r + \rho_0] = & a \left[1 + \frac{(\rho_1 r + \rho_0)^3}{6} + \frac{(\rho_1 r + \rho_0)^6}{180} + \frac{(\rho_1 r + \rho_0)^9}{12960} + \dots \right] - \\ & b \left[(\rho_1 r + \rho_0) + \frac{(\rho_1 r + \rho_0)^4}{12} + \frac{(\rho_1 r + \rho_0)^7}{504} + \frac{(\rho_1 r + \rho_0)^{10}}{45360} + \dots \right] \end{aligned} \quad (38)$$

with $a = 0.3550281, b = 0.2588194$.

Here we have studied the sensitivity of the order of polynomial approximation of the Airy's infinite series taking polynomial orders $r^3, r^4, r^6, r^7, r^9, r^{10}$ (as polynomial orders $r^{(2+3l)}$ with $l=0,1,2$ etc are absent in the Airy's function series). Further, it is found that the infinite limit of integration in calculating $\xi(y)$ and its derivatives makes the result divergent. Here, we take some reasonable cut-off limit r_0 of its integration. This will not sacrifice the nature and value of Airy's function and its derivatives, because, Airy's function falls very sharply (exponentially) and almost dies out with increasing r -value. (Table 1 and Fig. 1,2).

Table 1: Values of Airy's function for some small positive r .

r	Ai [r]	r	Ai [r]
0.1	0.329	2.5	0.016
0.5	0.232	3.0	0.007
1.0	0.135	3.5	0.003
1.5	0.072	4.0	0.0009
2.0	0.035	4.5	0.0003

We find that Airy's function value becomes negligibly small for $r > 5$. Also, the graph of normalization constants (N and N') versus the cut-off to upper limit (r_0) shows that N and N' value decreases with increase in r_0 (Fig. 3,4).

Also, the graph of ρ^2 vs r_0 (Fig 5) and C vs r_0 (Fig 6) confirms that, beyond $r_0 = 9$, ρ^2 and C values rise steeply as compared to the result of Table 2.

Table 2: Results of slope and curvature of $\xi(y)$ in different models and collaborations.

Model / collaboration	Value of slope	Value of curvature
Ref [6]	0.7936	0.0008
Le Youanc et al [25]	≥ 0.75	≥ 0.47
Skryme Model [26]	1.3	0.85
Neubert [27]	0.82 0.09	—
UK QCD Collab. [28]	0.83	—
CLEO [29,30]	1.67	—
BELLE [31]	1.35	—
HFAG [32]	1.17 ± 0.05	—
Huang [33]	1.35 ± 0.12	—

Upon this consideration, we have explored the $\xi(y)$ and its derivatives for different orders of polynomial approximation of Airy's function both for unperturbed wave function (Table 3) and total wave function (with relativistic effect) (Table 4), taking different cut-off values ranging from $r_0 = 5$ to $r_0 = 9$, for D meson taking $\alpha_s = 0.22$ [23].

Regarding sensitivity of the order of polynomial in infinite Airy's function, the result for ρ^2 and C do not differ much upon variation of order of polynomial in Airy's function. For a given Airy order, with increase in cut-off value, ρ^2 and C values increases steadily, whereas for

Table 3: Result with unperturbed wave-function with $r_0 = 5, 7, 9 \text{ GeV}^{-1}$.

O(r)in A_i	$r_0 = 5$			$r_0 = 7$			$r_0 = 9$		
	N	ρ^2	C	N	ρ^2	C	N	ρ^2	C
r^3	0.9307	0.5053	0.0872	0.8864	0.7595	0.2213	0.8656	0.9857	0.4581
r^4	1.0658	0.6712	0.1123	1.0121	0.8625	0.2195	1.0118	0.8658	0.2229
r^6	1.0374	0.6414	0.1066	0.9865	0.8343	0.2161	0.9835	0.8634	0.2468
r^7	1.0201	0.6221	0.1031	0.9720	0.8078	0.2071	0.9711	0.8160	0.2153
r^9	1.0235	0.6258	0.1037	0.9749	0.8128	0.2086	0.9737	0.8236	0.2196
r^{10}	1.0248	0.6274	0.1040	0.9760	0.8144	0.2090	0.9750	0.8242	0.2189

 Table 4: Result with total wave-function (with relativistic effect) taking $r_0 = 5, 7, 9 \text{ GeV}^{-1}$.

O(r)in A_i	$r_0 = 5$			$r_0 = 7$			$r_0 = 9$		
	N	ρ^2	C	N	ρ^2	C	N	ρ^2	C
r^3	1.5927	0.5149	0.0661	1.5658	0.6125	0.1269	1.4985	0.9681	0.4365
r^4	1.8653	0.6300	0.0869	1.8104	0.7942	0.1932	1.4375	2.4989	1.8879
r^6	1.8703	0.6432	0.0878	1.8162	0.8040	0.1921	1.5311	2.0925	1.4440
r^7	1.8471	0.6313	0.0858	1.7946	0.7899	0.1884	1.4769	2.2604	1.6355
r^9	1.8518	0.6340	0.0862	1.7989	0.7931	0.1892	1.4871	2.2294	1.5990
r^{10}	1.8531	0.6348	0.0863	1.8000	0.7940	0.1895	1.4847	2.2462	1.6169

a given cut-off value, ρ^2 and C values do not differ much upon variation of order of polynomial in Airy's function from r^4 to r^{10} . However, the results show closer resemblance to recent result of $\rho^2 = 1.17$ [32] for our Airy-order r^3 up to cut-off value $r_0 = 10$. For such specific range and order, our result shows improvement upon the result of ref [6]. At cut-off value higher than $r_0 = 9$, the results jumps to higher values than our expectations. In all calculations, we have taken care that the boundary condition of IWF ($\xi(1) = 1$) is maintained throughout. The variation of $\xi(y)$ with y for different cut-off value r^0 , with Airy order r^{10} , is shown in figure 7 and the variation of $\xi(y)$ with y for different Airy order at cut-off value $r_0 = 5$ is shown in Fig. 8. In 8, graphs of $\xi(y)$ vs y overlaps for Airy orders r^4 to r^{10} , whereas the graph for Airy order r^3 shows a slight deviation.

4 Conclusion and remarks:

We have found that cutting off the upper limit of integrations in $\xi(y)$ and its derivatives to some reasonable point does not upset the result, rather it almost conforms to the experimental expectations. Also, for each value of cut-off r_0 , we have considered the asymptotic form of the Airy's function taking limits of integration from r_0 to ∞ .

$$A_i[r]_{\text{asympt}} \sim \frac{\exp(-\frac{2}{3}r^{3/2})}{2\sqrt{\pi}r^{1/4}} \quad (39)$$

With this asymptotic form we have also calculated the derivatives of $\xi(y)$. Such analysis shows that very small values of ρ^2 and C result [Table 5], taking this asymptotic form of Airy's function. This, thus, also confirms the justification of taking some reasonable cut-off to upper limit

of integration in calculating $\xi(y)$ and its derivatives.

Table 5: Values of ρ^2 and C with asymptotic form of Airy's function.

r_0 value	ρ^2 (asymptotic)	C(asymptotic)
5	4.6×10^{-9}	1.6×10^{-9}
6	5.027×10^{-11}	2.464×10^{-11}
7	3.56×10^{-13}	2.345×10^{-13}
8	1.695×10^{-15}	7.028×10^{-15}
9	5.248×10^{-18}	6.597×10^{-15}
10	2.92×10^{-19}	2.78×10^{-19}

Plots of $\xi(y)$ versus y for different orders of Airy's function are also in agreement with our expectations [8]. The graph of $\xi(y)$ with y for different Airy's function order invariably starts at (1,1) and almost follows the same pattern and show very small deviation with change in Airy order. It confirms the fact that boundary condition for zero recoil ($\xi(1) = 1$) is maintained all through, with different polynomial orders of Airy's function and for different cut-off values.

Let us also comment on the result of Ref [6]. The result of ref [6], which is for Airy order r^3 , matches with our cut-off value $r_0 = 7.95 \text{ GeV}^{-1}$ for the same Airy order in our calculation. However, their wave function is not found to satisfy the zero recoil condition of IW function, $\xi(y) = 1$ for $y = 1$.

To conclude, we also study the compatibility our potential model with the recent results of Heavy Flavour Averaging Group [32]. Taking the result of ρ^2 , we fix the value of cut-off for different orders of polynomial in Airy function (Table 6). It indicates that, the range of cut-off value $r_0 = 7 \text{ GeV}^{-1}$ to $r_0 = 9 \text{ GeV}^{-1}$ matches the expectations of ref [32].

Table 6: Value of cut-off r_0 for different Airy order matching expectation of ρ^2 .

$\rho^2 = 1.17 \text{ GeV}^{-1}(\text{ref}[32])$	
Airy Order	r_0 value
r^3	8.896
r^4	7.915
r^6	7.975
r^7	7.942
r^9	7.939
r^{10}	7.932

Lastly, we would also like to comment on the limitation of the present formalism. The perturbed wave function $\Psi^1(r)$ has been calculated using Airy order r^3 , as in ref [6]. Although, the total wave function (eqn 22) contains infinite Airy series in terms of unperturbed wave function, this above mentioned limitation may have some effect on the values of the coefficients of the higher polynomial orders of Airy function in total wave function. Improvement of the formalism considering higher polynomial orders of Airy's function in $\Psi^1(r)$ is under consideration.

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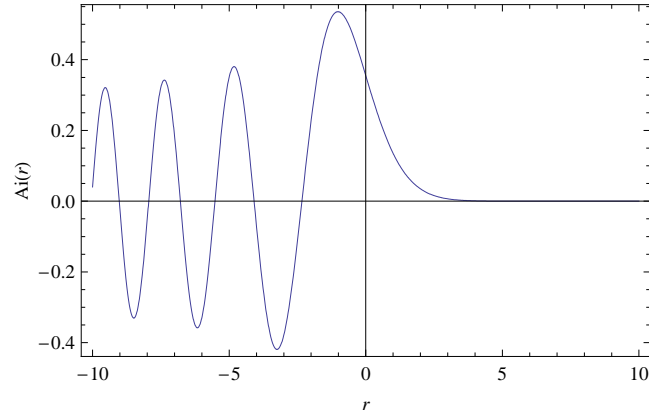


Figure 1: Variation of Airy's function with r

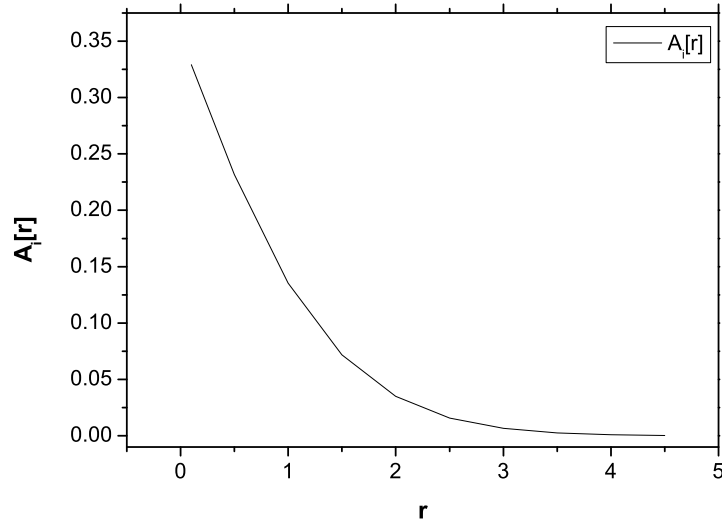


Figure 2: Variation of Airy's function with positive r

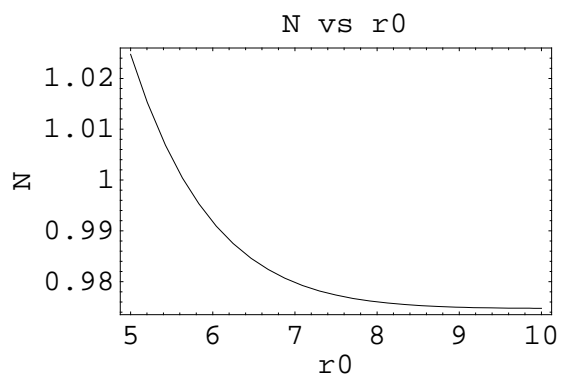


Figure 3: Variation of N with r_0 .

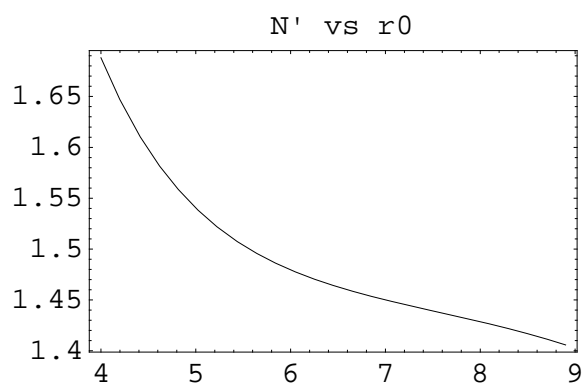


Figure 4: Variation of N' with r_0 .

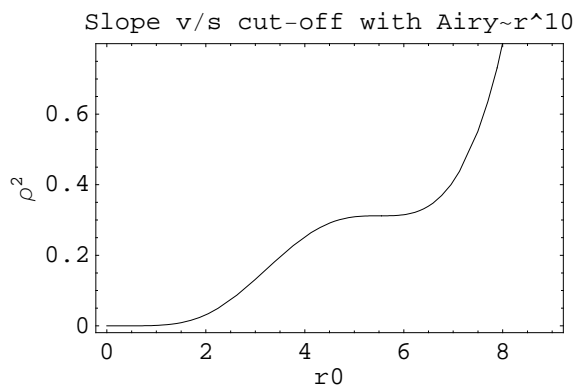


Figure 5: Variation of ρ^2 vs r_0 with Airy order r^{10} .

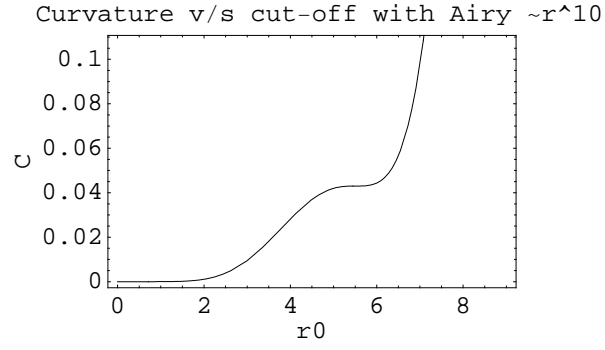


Figure 6: Variation of C vs r_0 with Airy order r_10 .

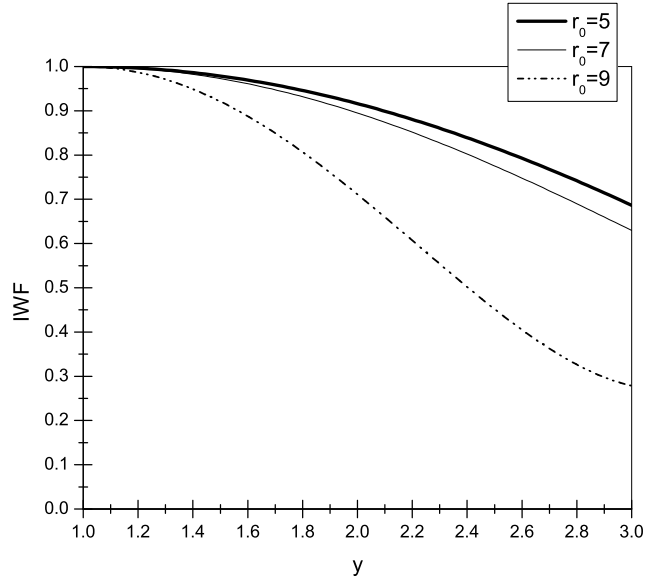


Figure 7: Variation of $\xi(y)$ with y taking different r_0 with Airy order r^{10} .

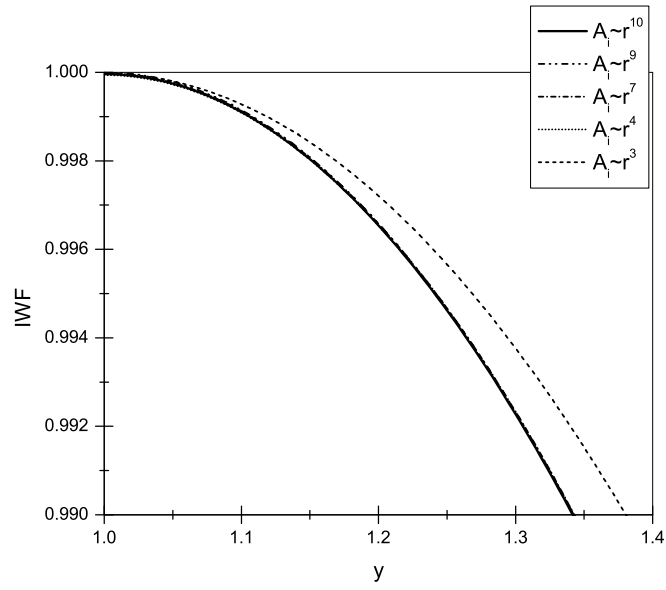


Figure 8: Variation of $\xi(y)$ with y taking different Airy order with $r_0 = 5$.